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p-Harmonic morphisms, biharmonic morphisms, and nonharmonic biharmonic maps

Ye-Lin Ou*

Department of Mathematics, The University of Oklahoma, 601 Elm. Av. Phsc. 423, Norman, OK 73019, USA

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Abstract

We study the transformation of a p-harmonic morphism into a q-harmonic morphism via biconformal change of the domain metric and/or conformal change of the codomain metric. As an application of p-harmonic morphisms, we characterize a twisted product among doubly twisted products and a warped product among twisted products using p-harmonicity of their projection maps. We describe those p-harmonic morphisms which are also biharmonic morphisms and give a complete classification of polynomial biharmonic morphisms between Euclidean spaces. Finally, we show that a horizontally homothetic harmonic morphism with harmonic energy density pulls back a nonharmonic biharmonic map to a nonharmonic biharmonic map and that totally geodesic immersing the target manifold of a nonharmonic biharmonic map into an ambient manifold produces a new nonharmonic biharmonic map. These methods are used to construct many examples of nontrivial biharmonic maps.

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* Tel.: +1 405 3256711; fax: +1 405 3257484.
E-mail address: ylou@ou.edu (Y.-L. Ou).

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1. Introduction

In this paper, we work in the category of smooth objects; so all manifolds, vector fields, and maps are assumed to be smooth unless otherwise stated.

1.1. p-Harmonic maps and morphisms

For $p > 1$, a *p-harmonic map* is a map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds such that $\varphi|\Omega$ is a critical point of the p-energy

$$E_p(\varphi, \Omega) = \frac{1}{p}\int_{\Omega} |\mathrm{d}\varphi|^p \,\mathrm{d}x$$

for every compact subset Ω of M. Locally, p-harmonic maps are solutions of the following systems of PDEs:

$$\tau_p(\varphi) = |\mathrm{d}\varphi|_g^{p-2}\tau_2(\varphi) + (p-2)|\mathrm{d}\varphi|_g^{p-3}\,\mathrm{d}\varphi(\mathrm{grad}_g|\mathrm{d}\varphi|_g) = 0,$$

where $\tau_2(\varphi) = \mathrm{Trace}_g\nabla\mathrm{d}\varphi$ denotes the tension field of φ. Note that when $|\mathrm{d}\varphi| \neq 0$, we can write

$$\tau_p(\varphi) = |\mathrm{d}\varphi|_g^{p-2}[\tau_2(\varphi) + (p-2)\mathrm{d}\varphi(\mathrm{grad}_g(\ln|\mathrm{d}\varphi|_g))]. \tag{1}$$

When $p = 2$, p-harmonic maps are well known to be harmonic maps which include geodesic, harmonic functions, and minimal isometric immersions as special cases.

A *p-harmonic morphism* is a map between Riemannian manifolds $\varphi : (M, g) \to (N, h)$ that preserves solutions of p-Laplace's equation in the sense that if $\triangle_p^N f = 0$, then $\triangle_p^M (f \circ \varphi) = 0$ for any function f defined (locally) on N. It is well known (see [8,16,21,27]) that a non-constant map between Riemannian manifolds is a p-harmonic morphism if and only if it is a horizontally weakly conformal p-harmonic map. A *horizontally weakly conformal* map $\varphi : (M, g) \to (N, h)$ generalizes the notion of a Riemannian submersion in that for any $x \in M$ at which $\mathrm{d}\varphi_x \neq 0$, the restriction $\mathrm{d}\varphi_x|_{H_x} : H_x \to T_{\varphi(x)}N$ is conformal and surjective, where the horizontal space H_x is the orthogonal complement of $V_x = \ker(\mathrm{d}\varphi_x)$ in T_xM. Thus it follows that there is a number $\lambda(x) \in (0, \infty)$ such that $h(\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y)) = \lambda^2(x)g(X, Y)$ for any $X, Y \in H_x$. Note that at the point $x \in M$ where $\mathrm{d}\varphi_x = 0$ we can let $\lambda(x) = 0$ to obtain a continuous function $\lambda : M \to R$ which is called the *dilation* of a horizontally weakly conformal map φ. A non-constant horizontally weakly conformal map φ is said to be *horizontally homothetic* if the gradient of $\lambda^2(x)$ is vertical, meaning that $X(\lambda^2) \equiv 0$ for any horizontal vector field X on M.

p-Harmonic maps of different p values have different regularity theory whilst p-Laplace operator of different p values have different applications in physics. Also, p-harmonic morphisms of different p values have different geometry as shown by the following theorem, which gives some interesting links among horizontal conformality, p-harmonicity and minimality of fibers of such maps.

Theorem 1.1. ([2,3,8,43]) *Let $m > n \geq 2$ and $\varphi : (M^m, g) \to (N^n, h)$ be a horizontally conformal submersion.*

(I) *If* $p = n$, *then* φ *is p-harmonic (hence a p-harmonic morphism) if and only if* $\{\varphi^{-1}(y)\}_{y\in N}$ *is a minimal foliation of* (M, g) *of codimension n.*

(II) *If* $p \neq n$, *then any two of the following conditions imply the other one*:

  (a) φ *is p-harmonic (hence a p-harmonic morphism),*

  (b) $\{\varphi^{-1}(y)\}_{y\in N}$ *is a minimal foliation of* (M, g) *of codimension n,*

  (c) φ *is horizontally homothetic.*

A 2-harmonic morphism is simply called a harmonic morphism examples of which include holomorphic functions, Hopf fibrations, and Riemannian submersions with minimal fibers. For a detailed background and developments of the study of harmonic morphisms, we recommend the recent book by Baird and Wood [6]. An updated bibliography for harmonic morphisms is available in Ref. [19]. For recent work on the classifications and constructions of p-harmonic morphisms, see Refs. [35–37].

1.2. Biharmonic maps and morphisms

A *biharmonic map* is a map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds such that $\varphi|\Omega$ is a critical point of the bienergy

$$E^2(\varphi, \Omega) = \frac{1}{2}\int_{\Omega} |\tau_2(\varphi)|^2 \, \mathrm{d}x$$

for every compact subset Ω of M, where $\tau_2(\varphi)$ is the tension field of φ. Jiang [22] derived the first and second variational formulas for bienergy functional, showing that φ is a biharmonic map if and only if its bitension field vanishes identically, i.e.,

$$\tau^2(\varphi) := -\triangle^{\varphi}(\tau_2(\varphi)) - \mathrm{Trace}_g R^N(\mathrm{d}\varphi, \tau_2(\varphi))\, \mathrm{d}\varphi = 0, \tag{2}$$

where

$$\triangle^{\varphi} = -\mathrm{Trace}_g(\nabla^{\varphi})^2 = -\mathrm{Trace}_g(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla^M})$$

is the Laplacian on sections of the pull-back bundle $\varphi^{-1}TN$ and R^N is the curvature operator of (N, h) defined by

$$R^N(X, Y)Z = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z.$$

Note that $\tau^2(\varphi) = -J^{\varphi}(\tau_2(\varphi))$, where J^{φ} is the Jacobi operator which plays an important role in the study of harmonic maps.

Clearly, any harmonic map is a biharmonic map, so it is interesting to study nonharmonic biharmonic maps. When the map is the canonical inclusion $\mathbf{i} : M \to \mathbb{R}^n$ of a submanifold of a Euclidean space, we denote by H the mean curvature vector and view it as a map $H : M \to \mathbb{R}^n$. Then, Chen [11] called the submanifold M a biharmonic submanifold if $\triangle H = (\triangle H^1, \ldots, \triangle H^n) = 0$, where $\triangle$ is the Beltrami–Laplace operator of the metric induced by $\mathbf{i}$. Noting that $\triangle H = \triangle(-\frac{1}{m}\triangle \mathbf{i}) = -\frac{1}{m}\triangle^2 \mathbf{i} = -\frac{1}{m}\tau^2(\mathbf{i})$ we see that M is a biharmonic submanifold of Euclidean space $\mathbb{R}^n$ if and only if $\tau^2(\mathbf{i}) = 0$, i.e., the inclusion map is a biharmonic map. It is well-known (see [14]) that an isometric immersion is minimal if and only if it is harmonic. So a minimal submanifold is trivially biharmonic. A conjecture made by Chen [11] states that any biharmonic submanifold of Euclidean space is minimal. Chen and Ishikawa [12] proved that there is no nonminimal (equivalently, nonharmonic) biharmonic surface in $\mathbb{R}^3$ so the conjecture is true in this case. Though the conjecture is also known to be true in some other cases (see e.g., [13,18,20]) it is still open for the general case. Caddeo et al.[9] obtained a classification of nonharmonic biharmonic submanifolds in S^3, whilst in Ref. [10], they shown that there is no nonharmonic biharmonic submanifold in hyperbolic 3-space $H^3(-1)$ and they also gave two ways to construct nonharmonic biharmonic submanifolds of S^n. Conformally deforming the domain or the target metric of a harmonic map $\varphi : (M, g) \to (N, h)$ so that φ becomes a nonharmonic biharmonic map was studied by Baird and Kamissoko [4] and Balmus [7].

A *biharmonic morphism* (see [34] for precise definition and background) is a map between Riemannian manifolds that pulls back local biharmonic functions to local biharmonic functions. These are characterized as a special subclass of horizontally weakly conformal biharmonic maps [30,34].

The rest of the paper is organized as follows. In Section 2, we study the transformation of a p-harmonic morphism into a q-harmonic morphism via biconformal change of the domain metric and/or conformal change of the codomain metric, and we characterize a twisted product and a warped product according to its projection map being a p-harmonic morphism for some particular value of p. Section 3 is devoted to the study of the relationship between p-harmonic morphisms and biharmonic morphisms. We describe those p-harmonic morphisms which are also biharmonic morphisms and give a complete classification of polynomial biharmonic morphisms between Euclidean spaces. Finally, in Section 4, we show that a horizontally homothetic harmonic morphism with harmonic energy density pulls back a nonharmonic biharmonic map to a nonharmonic biharmonic map and that totally geodesic immersing the target manifold of a nonharmonic biharmonic map into an ambient space produces a new nonharmonic biharmonic map. These are used to construct many examples of nontrivial biharmonic maps.

2. p-Harmonic morphisms and some applications

In this section, we first derive a formula for the p-tension field of a horizontally conformal submersion under a biconformal change of the domain metric and a conformal change of the codomain metric. We then use it to obtain the conditions under which a p-harmonic morphism is transformed into a q-harmonic morphism under biconformal and/or conformal change of metrics. We close the section by giving an application of p-harmonic morphisms in characterizing twisted and warped products.

Lemma 2.1. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a horizontally conformal submersion with dilation $\lambda : M \to (0, \infty)$. Let $g = g_{\mathrm{h}} + g_{\mathrm{v}}$ be the decomposition of the metric g into horizontal and vertical components. Let $\tilde{g}$ be a biconformal change of g and $\tilde{h}$ a conformal* *change of h given by*

$$(a)\quad \tilde{g} = \sigma^{-2} g_{\mathrm{h}} + \rho^{-2} g_{\mathrm{v}}, \qquad (b)\quad \tilde{h} = \bar{\nu}^{-2} h,$$

where $\sigma, \rho : M \to (0, \infty)$ *and* $\bar{\nu} : N \to (0, \infty)$ *are smooth functions. Denote* $\nu = \bar{\nu} \circ \varphi$. *Then, the p-tension field of the map* $\varphi : (M^m, \tilde{g}) \to (N^n, \tilde{h})$ *is given by*

$$\tilde{\tau}_p(\varphi) = n^{(p-2)/2} \lambda^{p-2} \sigma^p \nu^{2-p} \{\tau_2(\varphi) + \mathrm{d}\varphi(\mathrm{grad}\, \ln(\lambda^{p-2} \sigma^{p-n} \rho^{n-m} \nu^{n-p}))\}. \tag{3}$$

Proof. By Lemma 4.6.6 in [6], φ is a horizontally conformal submersion with respect to $\tilde{g}$ and $\tilde{h}$ on M and N with dilation $\tilde{\lambda} = \lambda\sigma\nu^{-1}$ and has tension field

$$\tilde{\tau}_2(\varphi) = \sigma^2 \{\tau_2(\varphi) + \mathrm{d}\varphi(\mathrm{grad}\, \ln(\sigma^{2-n} \rho^{n-m} \nu^{n-2}))\}. \tag{4}$$

A direct computation using the fact that φ is a horizontally conformal submersion gives

$$|\mathrm{d}\varphi|_{\tilde{g}}^{p-2} = n^{(p-2)/2} \lambda^{p-2} \sigma^{p-2} \nu^{2-p}. \tag{5}$$

On the other hand, one can check that the map φ with respect to the metrics $\tilde{g}$ and $\tilde{h}$ and the original map share the same horizontal and vertical spaces. Let $\{e_i\}$ (respectively $\{\tilde{e}_i\}$) be a local orthonormal frame of the horizontal distribution $\mathcal{H}$ with respect to metric g (respectively $\tilde{g}$). Then, $\tilde{e}_i = \sigma e_i$, and

$$\begin{aligned} \mathrm{d}\varphi(\mathrm{grad}_{\tilde{g}} f) &= \mathrm{d}\varphi((\mathrm{grad}_{\tilde{g}} f)_{\mathrm{h}} + (\mathrm{grad}_{\tilde{g}} f)_{\mathrm{v}}) = \mathrm{d}\varphi((\mathrm{grad}_{\tilde{g}} f)_{\mathrm{h}}) = \mathrm{d}\varphi\left(\sum_{i=1}^{i=n} (\tilde{e}_i f)\tilde{e}_i\right) \\ &= \sigma^2\, \mathrm{d}\varphi\left(\sum_{i=1}^{i=n} (e_i f) e_i\right) = \sigma^2 \mathrm{d}\varphi((\mathrm{grad}\, f)_{\mathrm{h}}) = \sigma^2 \mathrm{d}\varphi(\mathrm{grad}\, f) \end{aligned} \tag{6}$$

for any function f on M. Using (5) and (6) we have

$$(p-2)\mathrm{d}\varphi(\mathrm{grad}_{\tilde{g}}\, \ln|\mathrm{d}\varphi|_{\tilde{g}}) = \mathrm{d}\varphi(\mathrm{grad}_{\tilde{g}}\, \ln|\mathrm{d}\varphi|_{\tilde{g}}^{p-2}) = \sigma^2 \mathrm{d}\varphi(\mathrm{grad}\, \ln(\lambda^{p-2} \sigma^{p-2} \nu^{2-p})). \tag{7}$$

Substituting Eqs. (4), (5), and (7) into the p-tension field formula (1) we obtain (3), which completes the proof of the lemma. □

As an immediate consequence, we have the following corollary.

Corollary 2.2. $\varphi : (M^m, \tilde{g}) \to (N^n, \tilde{h})$ *is a p-harmonic morphism if and only if* $\tau_2(\varphi) + \mathrm{d}\varphi(\mathrm{grad}\, \ln(\lambda^{p-2} \sigma^{p-n} \rho^{n-m} \nu^{n-p})) = 0$.

Remark 1. Note that when $\nu = 1$, $\sigma = \rho = \alpha^{-1}$, Corollary 2.2 reduces to Lemma 5.1 in [8] where the authors constructed some nontrivial p-harmonic morphism via a conformal change of the domain metric.

Now we are ready to prove the following theorem which includes Proposition 4.6.8 in [6] as a special case.

Theorem 2.3. *For $p, q > 1$, let $\varphi : (M^m, g) \to (N^n, h)$ be a submersive p-harmonic morphism with dilation λ. Let $\tilde{g}$ be a biconformal change of g and $\tilde{h}$ a conformal change of h given by (a) and (b) in Lemma 2.1. Then, the map $\varphi : (M^m, \tilde{g}) \to (N^n, \tilde{h})$ is a q-harmonic morphism if and only if* $\mathrm{grad}(\lambda^{q-p}\sigma^{q-n}\rho^{n-m}\nu^{n-q})$ *is vertical; equivalently, the function $\lambda^{q-p}\sigma^{q-n}\rho^{n-m}\nu^{n-q}$ is constant along horizontal curves.*

Proof. Suppose that $\varphi : (M^m, g) \to (N^n, h)$ is a submersive p-harmonic morphism. Then, it is a p-harmonic horizontally conformal submersion. It follows that

$$\tau_p(\varphi) = |\mathrm{d}\varphi|^{p-2}\{\tau_2(\varphi) + (p-2)\mathrm{d}\varphi(\mathrm{grad}(\ln|\mathrm{d}\varphi|))\} = 0. \tag{8}$$

Noting that φ is a horizontally conformal submersion with dilation λ and $|\mathrm{d}\varphi|^2 = n\lambda^2 \neq 0$ we have, from (8),

$$\tau_2(\varphi) = \mathrm{d}\varphi(\mathrm{grad}(\ln\lambda^{2-p})). \tag{9}$$

Using (3) and (9) we obtain

$$\tilde{\tau}_q(\varphi) = n^{(q-2)/2}\lambda^{q-2}\sigma^q\nu^{2-q}\mathrm{d}\varphi(\mathrm{grad}\,\ln(\lambda^{q-p}\sigma^{q-n}\rho^{n-m}\nu^{n-q})). \tag{10}$$

Since φ is also a horizontally conformal submersion with respect to $\tilde{g}$ and $\tilde{h}$, we see from (10) that φ is a q-harmonic morphism with respect to $\tilde{g}$ and $\tilde{h}$ if and only if $\mathrm{d}\varphi(\mathrm{grad}\,\ln(\lambda^{q-p}\sigma^{q-n}\rho^{n-m}\nu^{n-q})) = 0$, which is equivalent to $\mathrm{d}\varphi(\mathrm{grad}\,(\lambda^{q-p}\sigma^{q-n}\rho^{n-m}\nu^{n-q})) = 0$. This means that $\mathrm{grad}(\lambda^{p-q}\sigma^{p-n}\rho^{n-m}\nu^{n-p})$ is vertical. Thus we obtain the theorem. □

Corollary 2.4. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a submersive p-harmonic morphism with dilation λ. Let $\tilde{h}$ be defined as in (b) in Lemma 2.1 and $\tilde{g} = \sigma^{-2}g$. Then, $\varphi : (M^m, \sigma^{-2}g) \to (N^n, \tilde{h})$ is a q-harmonic morphism if and only if* $\mathrm{grad}(\lambda^{q-p}\sigma^{q-m}\nu^{n-q})$ *is vertical. In particular, a submersive p-harmonic morphism remains a p-harmonic morphism under conformal changes $\sigma^{-2}g$ and $\tilde{h}$ of the domain and codomain metrics if and only if* $\mathrm{grad}(\sigma^{p-m}\nu^{n-p})$ *is vertical.*

Proof. Since $\tilde{g} = \sigma^{-2}g = \sigma^{-2}g_{\mathrm{h}} + \sigma^{-2}g_{\mathrm{v}}$, a direct application of Theorem 2.3 with $\rho = \sigma$ gives the first assertion. The second statement follows from the first one with $p = q$. □

Corollary 2.5. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a submersive p-harmonic morphism with dilation λ. Then, $\varphi : (M^m, \sigma^{-2}g) \to (N^n, h)$ is a q-harmonic morphism if and only if* $\mathrm{grad}(\lambda^{q-p}\sigma^{q-m})$ *is vertical. In particular, a submersive p-harmonic morphism remains a p-harmonic morphism under a non-horizontally homothetic conformal change $\sigma^{-2}g$ of the domain metric if and only if the original map is an m-harmonic morphism.*

Proof. Applying Corollary 2.4 with $\nu = 1$ gives the first statement. It follows that a submersive p-harmonic morphism remains a p-harmonic morphism under a conformal change $\sigma^{-2}g$ of the domain metric if and only if $\mathrm{grad}(\sigma^{p-m})$ is vertical, which is equivalent to $(p-m)\sigma^{p-m-1}\,\mathrm{d}\varphi(\mathrm{grad}\,\sigma) = 0$. The last statement follows since the conformal change of metric is non-horizontally homothetic, i.e., $\mathrm{d}\varphi(\mathrm{grad}\,\sigma) \neq 0$. □

Similarly, we have the following corollary.

Corollary 2.6. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a submersive p-harmonic morphism with dilation λ. Then, $\varphi : (M^m, g) \to (N^n, \beta^{-2}h)$ is a q-harmonic morphism if and only if* $\text{grad}(\lambda^{q-p}(\beta \circ \varphi)^{n-q})$ *is vertical. In particular, a submersive p-harmonic morphism remains a p-harmonic morphism under a non-homothetic conformal change $\beta^{-2}h$ of the codomain metric if and only if the original map is an n-harmonic morphism.*

Corollary 2.7. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a submersive p-harmonic morphism with dilation λ and $p \neq m$. Then, $\varphi : (M^m, \sigma^{-2}g) \to (N^n, h)$ is an m-harmonic morphism if and only if the original map is a horizontally homothetic submersion with minimal fibers and hence it is a p-harmonic morphism for any $p > 1$.*

Proof. Applying Corollary 2.5 with $q = m$ we conclude that the map $\varphi : (M^m, \sigma^{-2}g) \to (N^n, h)$ is an m-harmonic morphism if and only if $\text{grad}(\lambda^{m-p})$ is vertical which is equivalent to $\text{grad}(\lambda)$ being vertical since $p \neq m$. The latter implies that the original map is horizontally homothetic and hence, by [8], a p-harmonic morphism for any $p > 1$. It follows from Theorem 1.1 that φ has minimal fibers. □

Applying Corollary 2.6 with $q = n$, we have the following corollary.

Corollary 2.8. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a submersive p-harmonic morphism with dilation λ and $p \neq n$. Then, $\varphi : (M^m, \sigma^{-2}g) \to (N^n, h)$ is an n-harmonic morphism if and only if the original map is a horizontally homothetic submersion with minimal fibers and hence it is a p-harmonic morphism for any $p > 1$.*

Using Theorem 2.3, we have the following invariance of p-harmonic morphisms under biconformal changes of metric which generalizes the corresponding result for harmonic morphisms (i.e. $p = 2$ case) obtained in [31] (see also Corollary 4.6.10 in [6]).

Corollary 2.9. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a horizontally conformal submersion with dilation λ. Set $g_\sigma = \sigma^{-2}g_{\rm h} + \sigma^{2(n-p)/(m-n)}g_{\rm v}$. Then, $\varphi : (M^m, g_\sigma) \to (N^n, h)$ is a p-harmonic morphism if and only if the original map is a p-harmonic morphism.*

It was proved in [35] that if $\varphi : (M^m, g) \to (N^n, h)$ is a submersive p-harmonic morphism with dilation λ, and suppose that φ is not horizontally homothetic. Then, for any $q > 1$ and $q \neq m$, $\varphi : (M^m, |\mathrm{d}\varphi|^{2(p-q)/(m-q)}g) \to (N^n, h)$ is a q-harmonic morphism which is not horizontally homothetic. This provides a method to construct nontrivial q-harmonic morphisms from p-harmonic morphisms via a conformal change of the domain metric. The following corollary characterizes this as the only conformal change of metric of the form $|\mathrm{d}\varphi|^\alpha g$ that renders a nontrivial p-harmonic morphism a nontrivial q-harmonic morphism.

Corollary 2.10. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a submersive p-harmonic morphism with dilation λ. Suppose φ is not horizontally homothetic. Then, $\varphi : (M^m, |\mathrm{d}\varphi|^\alpha g) \to (N^n, h)$ is a q-harmonic morphism with $q \neq m$ if and only if $\alpha = \frac{2(p-q)}{m-q}$.*

Proof. Applying Corollary 2.5 with $\sigma = |\mathrm{d}\varphi|^{-\alpha/2}$ we see that $\varphi : (M^m, |\mathrm{d}\varphi|^\alpha g) \to (N^n, h)$ is a q-harmonic morphism if and only if $\mathrm{grad}(\lambda^{q-p}|\mathrm{d}\varphi|^{\alpha(m-q)/2})$ is vertical. Since φ is horizontally conformal submersion, $|\mathrm{d}\varphi| = \sqrt{n}\lambda$, we have

$$\mathrm{grad}(\lambda^{q-p}|\mathrm{d}\varphi|^{\alpha(m-q)/2}) = \mathrm{grad}\,(C\lambda^{[\alpha(m-q)/2+(q-p)]}), \tag{11}$$

where C is a nonzero constant. Note that $\mathrm{grad}(C\lambda^{[\alpha(m-q)/2+(q-p)]})$ is vertical if and only if $\mathrm{grad}\,\lambda$ is vertical or else $\alpha(m-q)/2 + (q-p) = 0$, i.e. $\alpha = \frac{2(p-q)}{m-q}$. By assumption, φ is not horizontally homothetic, i.e. $\mathrm{grad}\,\lambda$ is not vertical, thus we obtain the corollary. □

Recall that the doubly twisted product of Riemannian manifolds (M, g) and (N, h) with twisting functions $\alpha,\ \beta : M \times N \to (0, \infty)$ is referred to the Riemannian manifold $(M \times N, \alpha^2 g + \beta^2 h)$ which is denoted by ${}_{\alpha^2}M \times_{\beta^2} N$. When $\alpha \equiv 1$ we have $M \times_{\beta^2} N$, a twisted product with twisting function $\beta(x, y)$. When $\alpha \equiv 1$ and β depends only on the points in M we have a warped product with the warping function $\beta(x)$. For more study on the geometry of doubly twisted products we refer to [39]. For curvature conditions for a twisted product to be a warped product, see [15]. Svensson in [42] characterizes warped products as special harmonic morphisms. As a generalization, Ou [35] gives a characterization of twisted products as special n-harmonic morphisms. Our next theorem characterizes a twisted product among doubly twisted products and a warped product among twisted products using p-harmonicity of their projection maps. In proving the theorem, we need the following lemma.

Lemma 2.11. *The projection ${}_{\alpha^2}M^m \times_{\beta^2} N^n \to (N^n, h), \varphi(x, y) = y$, of a doubly twisted product onto its second factor is a p-harmonic morphism if and only if $\alpha^m \beta^{n-p} = f(x)$ for some function $f : M \to (0, \infty)$.*

Proof. Consider the projection of the Riemannian product

$$(M^m \times N^n, G = g + h) \to (N, h), \quad \varphi(x, y) = y. \tag{12}$$

It is a Riemannian submersion with totally geodesic fibers and hence a harmonic morphism with dilation $\lambda = 1$. Note that the horizontal space at the point (x, y) can be identified with $T_y N$ and hence $G_{\mathrm{h}} = h$, $G_{\mathrm{v}} = g$. Applying Theorem 2.3 with $p = 2, q = p, \lambda = 1, \nu = 1$, $\sigma = \beta^{-1}$ and $\rho = \alpha^{-1}$ we conclude that the projection ${}_{\alpha^2}M^m \times_{\beta^2} N^n \to (N^n, h), \varphi(x, y) = y$, of a doubly twisted product onto its second factor is a p-harmonic morphism if and only if $\mathrm{grad}_G(\alpha^m \beta^{n-p})$ is vertical. This, in local coordinates, is equivalent to

$$h^{ij} \frac{\partial}{\partial y^i}(\alpha^m \beta^{n-p}) \frac{\partial}{\partial y^j} = 0.$$

It follows that

$$h^{ij}\frac{\partial}{\partial y^i}(\alpha^m\beta^{n-p}) = 0 \tag{13}$$

for any $j = 1, \ldots, n$. Since the metric h is positive definite we see from Eq. (13) that $\frac{\partial}{\partial y^i}(\alpha^m\beta^{n-p}) = 0$ for any i hence the function $\alpha^m\beta^{n-p}$ does not depend on the points in N. Thus we obtain the lemma. □

Theorem 2.12. *Let $\alpha, \beta : M^m \times N^n \to (0, \infty)$ be two functions. Then,*

(1) *the projection $\varphi :_{\alpha^2} M^m \times_{\beta^2} N^n \to (N^n, h), \varphi(x, y) = y$, of a doubly twisted product onto its second factor is an n-harmonic morphism if and only if $_{\alpha^2}M^m \times_{\beta^2} N^n$ can be written as a twisted product;*
(2) *the projection $\varphi : M^m \times_{\beta^2} N^n \to (N^n, h), \varphi(x, y) = y$, of a twisted product onto its second factor is a p-harmonic morphism with $p \neq n$ if and only if $M^m \times_{\beta^2} N^n$ can be written as a warped product;*
(3) *the projection $\varphi : M^m \times_{\beta^2} N^n \to (M, g), \varphi(x, y) = x$, of a twisted product onto its first factor is a p-harmonic morphism if and only if $M^m \times_{\beta^2} N^n$ can be written as a Riemannian product.*

Proof. For statement (1), we know from [[35], Proposition 2.11] that the projection of a twisted product onto its second factor is an n-harmonic morphism. Now suppose the projection $\varphi :_{\alpha^2} M^m \times_{\beta^2} N^n \to (N^n, h), \varphi(x, y) = y$, of a doubly twisted product onto its second factor is an n-harmonic morphism. Then, by Lemma 2.11, $\alpha = (1/f(x))^{1/m}$ for some function $f : M \to (0, \infty)$. It follows that the doubly twisted product $_{\alpha^2}M^m \times_{\beta^2} N^n$ can be written as a twisted product of $(M^m, \bar{g})$ and (N^n, h) with the twisting function β, where $\bar{g} = \alpha^2(x)g$ is a metric on M conformal to g.

For statement (2), we know from [42] (see also [6], Proposition 2.4.26) that the projection of a warped product onto its second factor is a horizontally homothetic harmonic morphism hence a p-harmonic morphism for any $p > 1$ by [8]. Conversely, suppose the projection $\varphi : M^m \times_{\beta^2} N^n \to (N^n, h), \varphi(x, y) = y$, of a twisted product onto its second factor is a p-harmonic morphism with $p \neq n$. Using Lemma 2.11 with $\alpha = 1$ and the fact that $p \neq n$ we conclude that the twisting function $\beta = (f(x))^{1/(n-p)}$ for some function $f : M \to (0, \infty)$. It follows that β depends only on the points in M, so the twisted product $M^m \times_{\beta^2} N^n$ is in fact a warped product.

To prove (3), we note that the horizontal and vertical distributions of φ are $\mathcal{H} = TM$ and $\mathcal{V} = TN$, respectively. Let $G = g + \beta^2 h$, then $G_{\rm h} = g$ and $G_{\rm v} = \beta^2 h$. Since the projection $\varphi : M^m \times N^n \to (M, g), \varphi(x, y) = x$ is a Riemannian submersion with totally geodesic fibers, it is a harmonic morphism with dilation $\lambda = 1$. Noting that $M^m \times_{\beta^2} N^n$ is isometric to $_{\beta^2}N^n \times M^m$, we can apply Theorem 2.3 with $p = 2, q = p, \lambda = \nu = 1, \sigma = 1$ and $\rho = \beta^{-1}$ to conclude that the projection $\varphi : M^m \times_{\beta^2} N^n \to (M, g), \varphi(x, y) = x$, of a twisted product onto its first factor is a p-harmonic morphism if and only if $\mathrm{grad}_G(\beta^m)$ is vertical. This, together with the fact that the horizontal distribution is integrable, implies that the twisting function β does not depend on the points in M. Thus, the twisted product $M^m \times_{\beta^2} N^n$ can be written as a Riemannian product of (M^m, g) and $(N^n, \bar{h})$, where $\bar{h}$ is a metric conformal to h on N. This ends the proof of the theorem. □

From (3) of Theorem 2.12 we can easily deduce the following corollary.

Corollary 2.13. *For any $p > 1$, the projection $\varphi : M^m \times_{\beta^2} N^n \to (M, g)$, $\varphi(x, y) = x$, of a warped product onto its first factor is a p-harmonic morphism if and only if β is a constant and hence $M^m \times_{\beta^2} N^n$ is in fact a Riemannian product up to a homothety.*

Remark 2. Yun proved in [[44], Theorem 2.4] that the projection of a warped product onto its first factor is harmonic (hence a harmonic morphism) if and only if the warping function is a constant. Clearly, Corollary 2.13 includes Yun's result as a special case.

3. *p*-Harmonic and biharmonic morphisms

In this section, we characterize those p-harmonic morphisms which are also biharmonic morphisms and give some examples of such maps which includes harmonic Riemannian submersions and projections of some warped products as subclasses. These will be used to construct nonharmonic biharmonic maps in the next section. We close the section by giving a complete classification of polynomial biharmonic morphisms between Euclidean spaces.

Theorem 3.1. *For $p \neq 4$, a submersive p-harmonic morphism $\varphi : (M^m, g) \to (N^n, h)$ is also a biharmonic morphism if and only if φ is a horizontally homothetic harmonic morphism with harmonic energy density, i.e., $\triangle_g(n\lambda^2/2) = 0$.*

Proof. If φ is a horizontally homothetic harmonic morphism, then it is a submersion by [17], and it is a p-harmonic morphism for any $p > 1$ by [8]. If, in addition, φ has harmonic energy density, then it is also a biharmonic morphism by Theorem 3.8 in [34]. Thus we obtain the "if part" of the theorem. For the "only if part", suppose φ is a submersive p-harmonic morphism. Then, it is a p-harmonic horizontally conformal submersion with dilation λ such that $|d\varphi|^2 = n\lambda^2$. Using (1) we have

$$\tau_2(\varphi) + (p-2)d\varphi(\mathrm{grad}_g(\ln\lambda)) = 0. \quad (14)$$

On the other hand, if φ is also a biharmonic morphism, then, by Theorem 4.1 in [30], we have

$$\lambda^2\tau_2(\varphi) + d\varphi(\mathrm{grad}_g\,\lambda^2) = 0,$$

which can be written as

$$\tau_2(\varphi) + 2d\varphi(\mathrm{grad}_g(\ln\lambda)) = 0. \quad (15)$$

It follows from (14) and (15) that

$$(p-4)d\varphi(\mathrm{grad}_g(\ln\lambda)) = 0. \quad (16)$$

Thus, if $p \neq 4$, then $d\varphi(\text{grad}_g(\ln\lambda)) = 0$, which means that φ is a horizontally homothetic submersion. It follows from [8] that φ is a p-harmonic morphism for any $p > 1$ and in particular a horizontally homothetic harmonic morphism. It follows then from Theorem 3.8 of [34] that φ must have harmonic energy density. Thus, we complete the proof of the theorem. □

Proposition 3.2. *The radial projection $\varphi : \mathbb{R}^m \setminus \{0\} \to S^{m-1}, \varphi(x) = x/|x|$ is a biharmonic morphism if and only if $m = 4$.*

Proof. It is known (see, e.g. [6]) that the radial projection $\varphi : \mathbb{R}^m \setminus \{0\} \to S^{m-1}, \varphi(x) = x/|x|$ is a horizontally homothetic harmonic morphism with dilation $\lambda(x) = 1/|x|$. By Theorem 3.1, φ is also a biharmonic morphism if and only if $\triangle^{\mathbb{R}^m}\lambda^2 = \triangle^{\mathbb{R}^m}(|x|^{-2}) = 0$. Let $f : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}$ be the function given by $f(x) = |x|^\alpha$. Then, a direct computation gives $\triangle^{\mathbb{R}^m} f = \triangle^{\mathbb{R}^m}(|x|^\alpha) = \alpha(\alpha - 2 + m)|x|^{\alpha-2}$. It follows that $|x|^\alpha$ is a harmonic function in $\mathbb{R}^m \setminus \{0\}$ if and only if $\alpha = 2 - m$. In particular, $\lambda^2 = |x|^{-2}$ is a harmonic function on $\mathbb{R}^m \setminus \{0\}$ if and only if $m = 4$. Thus, the proposition follows. □

Remark 3. We remark that a four-dimensional domain seems to have a mysterious link to biharmonicity since in [4] it was proved that inversion in the unit sphere $\sigma : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}, \sigma(x) = x/|x|^2$ is a biharmonic nonharmonic map if and only if $m = 4$.

The following proposition provides a class of horizontally homothetic harmonic morphisms with harmonic energy density and hence a class of maps which are both harmonic and biharmonic morphisms.

Proposition 3.3. *The projection $\varphi : M^m \times_{\beta^2} N^n \to (N^n, h), \varphi(x, y) = y$, of a warped product onto its second factor is a biharmonic morphism if and only if $1/\beta^2$ is a harmonic function on M.*

Proof. Note that the projection $M^m \times_{\beta^2} N^n \to (N^n, h), \varphi(x, y) = y$, of a warped product onto its second factor is a horizontally homothetic harmonic morphism with dilation $\lambda = 1/\beta$ [42] (see also [6]). By Theorem 3.1, φ is a biharmonic morphism if and only if $\triangle_{g+\beta^2 h}\lambda^2 = 0$. Since $\lambda = 1/\beta$ is a function defined on M, one can easily check that $\triangle_{g+\beta^2 h}\lambda^2 = \triangle_g\lambda^2 = \triangle_g(1/\beta^2)$, from which the corollary follows. □

Theorem 3.4. *For $m > n \geq 2$, a polynomial map (i.e. a map whose component functions are polynomials) $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is a biharmonic morphism if and only if it is a composition of an orthogonal projection followed by a homothety.*

Proof. It is well known (see, e.g. [6]) that the composition of an orthogonal projection followed by a homothety is a horizontally homothetic harmonic morphism with constant energy density. Thus, by Theorem 3.1, it is also a biharmonic morphism. Conversely, suppose φ is a biharmonic morphism, then, by [34], it is a special horizontally weakly conformal biharmonic map. Since φ is assumed to be a polynomial map, it is harmonic by a theorem in [1], which states that a horizontally weakly conformal polynomial map between Euclidean spaces is harmonic. It follows that φ is a harmonic morphism since it is a horizontally weakly conformal harmonic map [16,21]. By Theorem 3.8 in [34], φ is a horizontally homothetic harmonic morphism with harmonic energy density being both a harmonic morphism and a biharmonic morphism. It follows from [17] that φ is a submersion since it is a nonconstant horizontally homothetic harmonic map. Finally, using the classification of horizontally homothetic submersion $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ ([36], Theorem 2.7) we conclude that φ is a composition of an orthogonal projection followed by a homothety. □

Remark 4. Note that there are many polynomial harmonic morphisms between Euclidean spaces (for classifications of quadratic harmonic morphisms, see [33,38]). However, as indicated by Theorem 3.4, the only polynomial biharmonic morphism between Euclidean spaces is a composition of an orthogonal projection followed by a homothety. This is also true for p-harmonic morphism with $p \neq 2$ (see [36], Theorem 2.8).

4. Some constructions of nonharmonic biharmonic maps

In this section, we prove that a horizontally homothetic harmonic morphism with harmonic energy density pulls back nonharmonic biharmonic maps to nonharmonic biharmonic maps. We also show that totally geodesic immersing the target manifold of a nonharmonic biharmonic map into an ambient manifold produces a new nonharmonic biharmonic map. We use these two methods to construct many examples of nonharmonic biharmonic maps from or into the standard spheres.

Theorem 4.1. *Let $\varphi : (M, g) \to (N, h)$ be a nonconstant horizontally homothetic harmonic morphism with harmonic energy density, and let $\psi : (N, h) \to (Q, k)$ be a map. Then, the composition $\psi \circ \varphi : (M, g) \to (Q, k)$ is a nonharmonic biharmonic map if and only if ψ is nonharmonic biharmonic on the open subset $\varphi(M) \subseteq N$.*

Proof. Since φ is a harmonic morphism, we have (see [16,21])

$$\tau_2(\psi \circ \varphi) = \lambda^2 \tau_2(\psi) \circ \varphi. \tag{17}$$

On the other hand, by Theorem 3.1, φ is also a biharmonic morphism. It follows from Proposition 4.5 in [30] that

$$\tau^2(\psi \circ \varphi) = \lambda^4 \tau^2(\psi) \circ \varphi, \tag{18}$$

where $\tau^2(\psi)$ denotes the bitension field of ψ. From (17) and (18) we conclude that $\psi \circ \varphi$ is nonharmonic biharmonic if and only if ψ is nonharmonic biharmonic on $\varphi(M) \subseteq N$. Since φ is a harmonic morphism it is an open mapping [16] and hence $\varphi(M) \subseteq N$ is an open subset. Thus, we obtain the theorem. □

Noting that a harmonic Riemannian submersion is a horizontally homothetic harmonic morphism with constant hence harmonic energy density we have the following corollary.

Corollary 4.2. *Let $\varphi : (M, g) \to (N, h)$ be a harmonic Riemannian submersion, and let $\psi : (N, h) \to (Q, k)$ be a map. Then, the composition $\psi \circ \varphi : (M, g) \to (Q, k)$ is a non-*

harmonic biharmonic map if and only if ψ is nonharmonic biharmonic map on the open subset $\varphi(M) \subseteq N$.

Remark 5.

(1) Let $\varphi : (M, g) \to (N, h)$ be a harmonic Riemannian submersion, and let $\mathbf{i} : (N, h) \to (N, k)$ be the identity map. It was proved in [[7], Proposition 2.1 and Corollary 2.2] that the composition $\mathbf{i} \circ \varphi : (M, g) \to (N, k)$ is a nonharmonic biharmonic map if and only if $\mathbf{i}$ is a nonharmonic biharmonic map on the open subset $\varphi(M) \subseteq N$. Clearly, this is a very special case of Theorem 4.1.

(2) Let $S^n(a) = S^n(a) \times \{b\} = \{(x^1, \ldots, x^{n+1}, b) | \sum_{i=1}^{n+1} (x^i)^2 = a^2, a \in (0, 1), a^2 + b^2 = 1\}$, and $\mathbf{i} : S^n(a) \to S^{n+1}$ be the canonical inclusion. Then, it was proved in [9] that $\mathbf{i}$ is a nonharmonic biharmonic map if and only $a = 1/\sqrt{2}$, and $b = \pm 1/\sqrt{2}$. Let $\varphi : (M, g) \to S^n(a)$ be a harmonic Riemannian submersion. Then, it was proved in [[32], Theorem 2.1] that $\mathbf{i} \circ \varphi : (M, g) \to S^{n+1}$ is a nonharmonic biharmonic map if and only if $a = 1/\sqrt{2}$ and $b = \pm 1/\sqrt{2}$. Note that when φ is onto or M is compact, the same results follows immediately from Corollary 4.2.

(3) For other construction of nonharmonic biharmonic maps using composition of a harmonic map and an inclusion see [28] (also [29]) where the authors prove that for a nonconstant map $\varphi : (M, g) \to S^n(\frac{r}{\sqrt{2}})$, the map $\mathbf{i} \circ \varphi : (M, g) \to S^{n+1}(r)$ is nonharmonic biharmonic if and only φ is harmonic with harmonic energy density.

Combining Theorem 4.1 and Proposition 3.3, we have the following corollary.

Corollary 4.3. *Let $f : (M, g) \to (0, \infty)$ be a harmonic function and $\psi : (N, h) \to (Q, k)$ be a nonharmonic biharmonic map. Then, the composition of the projection $p_2 : M \times_{\frac{1}{f}} N \to N$ of the warped product followed by ψ is a nonharmonic biharmonic map.*

Recall that the second fundamental form $\nabla \, \mathrm{d}\varphi \in \Gamma(T^*M \otimes T^*M \otimes \varphi^{-1}TN)$ of a map $\varphi : (M, g) \to (N, h)$ is defined by $\nabla \, \mathrm{d}\varphi(X, Y) = \nabla^{\varphi}_X(\mathrm{d}\varphi(Y)) - \mathrm{d}\varphi(\nabla^M_X Y), \forall X, Y \in \Gamma(TM)$. A map is totally geodesic if its second fundamental form vanishes identically. It is well known (see, e.g. [14]) that post-composition of a totally geodesic map to a harmonic map yields a harmonic map. The following proposition can be viewed as generalizations of this and it provides a method to construct p-harmonic maps and nonharmonic biharmonic maps.

Proposition 4.4. *Let $\psi : N \to (Q, k)$ be the inclusion map of a totally geodesic submanifold, and let $\varphi : (M, g) \to (N, h = \psi^* k)$ be a map into the submanifold. Then,*

(i) *the map $\psi \circ \varphi$ into the ambient space (Q, k) is a p-harmonic map if and only if φ is a p-harmonic map.*

(ii) *the map $\psi \circ \varphi$ into the ambient space (Q, k) is a nonharmonic biharmonic map if and only if φ is a nonharmonic biharmonic map.*

Proof. It is well known [14] (see also [6], Proposition 3.3.12) that the second fundamental form and the tension field of the composition map verify the following identities:

$$\nabla \mathrm{d}(\psi \circ \varphi) = \mathrm{d}\psi(\nabla \mathrm{d}\varphi) + \nabla \mathrm{d}\psi(\mathrm{d}\varphi, \mathrm{d}\varphi), \tag{19}$$

$$\tau_2(\psi \circ \varphi) = \mathrm{d}\psi(\tau_2(\varphi)) + \mathrm{Trace}_g \nabla \mathrm{d}\psi(\mathrm{d}\varphi, \mathrm{d}\varphi). \tag{20}$$

Recall that if $\psi : N \to (Q, k)$ is the inclusion map of a submanifold, or more generally, an isometric immersion, then we can have an orthogonal decomposition of the vector bundle $\psi^{-1}TQ = \tau N \oplus \nu N$ into the tangent and normal bundles. We use $\mathrm{d}\psi$ to identify TN with its image τN in $\psi^{-1}TQ$. Then, for any $X, Y \in \Gamma(TN)$ we have $\nabla^{\psi}_X(\mathrm{d}\psi(Y)) = \nabla^{Q}_X Y$, whereas $\mathrm{d}\psi(\nabla^N_X Y)$ equals the tangential component of $\nabla^Q_X Y$. It follows that $\nabla \mathrm{d}\psi(X, Y)$ equals the normal component of $\nabla^Q_X Y$. This, by definition, is the second fundamental form $B(X, Y)$ of the immersed submanifold $\psi(N)$ in Q (see [23], Chapter 7). Thus, the second fundamental form of an isometric immersion $\psi : N \to (Q, k)$ (as a map) equals the second fundamental form of the immersed submaniflod $\psi(N) \subseteq Q$ (see also [6], Example 3.2.3). Therefore, the inclusion map of a totally geodesic submanifold is a totally geodesic map. This, together with (20), gives

$$\tau_2(\psi \circ \varphi) = \mathrm{d}\psi(\tau_2(\varphi)). \tag{21}$$

On the other hand, using local coordinates, we have

$$\begin{aligned} |\mathrm{d}(\psi \circ \varphi)|^2 &= g^{ij} \frac{\partial(\psi \circ \varphi)^\alpha}{\partial x^i} \frac{\partial(\psi \circ \varphi)^\beta}{\partial x^j} k_{\alpha\beta} = g^{ij} \frac{\partial \varphi^A}{\partial x^i} \frac{\partial \varphi^B}{\partial x^j} \frac{\partial \psi^\alpha}{\partial y^A} \frac{\partial \psi^\beta}{\partial y^B} k_{\alpha\beta} \\ &= g^{ij} \frac{\partial \varphi^A}{\partial x^i} \frac{\partial \varphi^B}{\partial x^j} h_{AB} = |\mathrm{d}\varphi|^2, \end{aligned} \tag{22}$$

where the third equality was obtained by using that fact that ψ is an isometric immersion and hence $\frac{\partial \psi^\alpha}{\partial y^A} \frac{\partial \psi^\beta}{\partial y^B} k_{\alpha\beta} = h_{AB}$. Using (21), (22) and the definition of p-tension field we have

$$\begin{aligned} \tau_p(\psi \circ \varphi) &= |\mathrm{d}\varphi|_g{}^{p-2} \mathrm{d}\psi(\tau_2(\varphi)) + (p-2)|\mathrm{d}\varphi|_g{}^{p-3} (\mathrm{d}\psi) \circ \mathrm{d}\varphi(\mathrm{grad}_g |\mathrm{d}\varphi|_g) \\ &= \mathrm{d}\psi(\tau_p(\varphi)), \end{aligned} \tag{23}$$

which gives statement (i).

For statement (ii), we note that the inclusion map ψ is a totally geodesic map, so we can use a theorem in [41] to have the bitension field of the composition $\tau^2(\psi \circ \varphi) = \mathrm{d}\psi(\tau^2(\varphi))$. It follows that $\psi \circ \varphi$ is biharmonic if and only if φ is biharmonic. On the other hand, by (21) we see that $\psi \circ \varphi$ is nonharmonic if and only if φ is nonharmonic since ψ is an immersion. This completes the proof of the proposition. □

An interesting problem in the study of harmonic maps concerns the existence and nonexistence of harmonic maps from standard spheres into a manifold. For example, in their series of papers, Lin and Wang [24–26] study approximable harmonic maps based on the existence and nonexistence of nonconstant harmonic maps $S^2 \to N$ called harmonic 2-spheres. One of their conjectures is (see [24]): any weakly harmonic map of finite energy from M^m into N is smooth if there are no harmonic spheres S^k in N for $2 \leq k \leq m-1$. A well-known theorem of Sacks and Uhlenbeck [40] guarantees the existence of harmonic 2-sphere in N if the universal covering space of N is not contractible. However, Sacks and Uhlenbeck's technique, as they pointed out in their paper, does not extend to give the existence of higher-dimensional harmonic spheres. In [35], it was shown that there exists harmonic 3-sphere in N when the universal covering space of N is not contractible. To the author's knowledge (see also remark in [24]), there has not been any general statement concerning the existence of higher dimensional harmonic spheres in the literature besides the above-mentioned result. By contrast, the following theorem shows that nonharmonic biharmonic spheres are abundant.

Theorem 4.5. *For any $n \geq 2$, the standard sphere S^n admits a nonharmonic (equivalently, non-minimal) biharmonic homothetic immersion into S^{n+k} for $k \geq 1$.*

Proof. For any $n \geq 2$, let $\varphi : S^n \to S^n(\frac{1}{\sqrt{2}})$, $\varphi(x) = x/\sqrt{2}$, denote the standard homothety. It is easy to see that φ is a horizontally homothetic harmonic morphism with constant energy density hence it is also a biharmonic morphism by Theorem 3.1. It follows from Theorem 4.1 that the map $\mathbf{i} \circ \varphi : S^n \to S^n(\frac{1}{\sqrt{2}}) \cong S^n(\frac{1}{\sqrt{2}}) \times \{\frac{1}{\sqrt{2}}\} \to S^{n+1}$, where $\mathbf{i} : S^n(\frac{1}{\sqrt{2}}) \times \{\frac{1}{\sqrt{2}}\} \to S^{n+1}$ denotes the canonical inclusion, is a nonharmonic biharmonic map since the inclusion $\mathbf{i}$ is a nonharmonic biharmonic map [9]. Let $\psi : S^{n+1} \to S^{n+k}$ be the totally geodesic inclusion which maps S^{n+1} onto the equator of S^{n+k}. Then, using (ii) of Proposition 4.4 we see that $\psi \circ \mathbf{i} \circ \varphi$ gives a nonharmonic biharmonic homothetic immersion of S^n into S^{n+k} for any $k \geq 1$. □

The following corollary produces many examples of nonharmonic biharmonic maps into spheres.

Corollary 4.6.

(i) *For any $n \geq 3$, there exists a nonharmonic biharmonic map $\varphi : S^2 \to S^n$.*
(ii) *For any $n \geq 3$, there exists a nonharmonic biharmonic map $\varphi : S^3 \to S^n$.*
(iii) *For any $n \geq 5$, there exists a nonharmonic biharmonic map $\varphi : S^7 \to S^n$.*
(iv) *For any $n \geq 9$, there exists a nonharmonic biharmonic map $\varphi : S^{15} \to S^n$.*
(v) *For any $n \geq 3$, there exists a nonharmonic biharmonic map $\varphi : S^3 \times S^3 \to S^n$.*
(vi) *For any $n \geq 5$, there exists a nonharmonic biharmonic map $\varphi : S^7 \times S^7 \to S^n$.*
(vii) *For any $n \geq 3$, there exists a nonharmonic biharmonic map $\varphi : \mathbb{R}^4 \setminus \{0\} \to S^n$.*

Proof. (i) follows from Theorem 4.5 with $n = 2$. For (ii),(iii) and (iv), let $h : S^{2n-1} \to S^n$ ($n = 2, 4, 8$) be the Hopf fibration which is well-known (see, e.g. [6]) to be a horizontally homothetic harmonic morphism with constant energy density. It follows from Theorem 3.1 that it is also a biharmonic morphism. Let $\varphi : S^n \to S^{n+k}$ be the nonharmonic biharmonic homothetic immersion defined in Theorem 4.5. Then, $\varphi \circ h$ gives the required nonharmonic biharmonic map. For (v) and (vi), let $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with $n = 4, 8$, and $F(x, y) = xy$ denote the standard multiplication in the real algebra of quaterionic or Cayley numbers. It is proved in [5] that this harmonic morphism restricts to a harmonic morphism $f = F|_{S^{n-1} \times S^{n-1}} : S^{n-1} \times S^{n-1} \to S^{n-1}$, where the target sphere is given the standard metric

g_0 and the domain manifold is given the product metric $g_0 \otimes g_0$. It is easily seen that the map f is non-trivial, i.e., it is not one of the projections to the factor, and we can check that it has dilation $\lambda = \sqrt{2}$, hence it is a horizontally homothetic harmonic morphism with harmonic energy density. Therefore, it is also a biharmonic morphism by Theorem 3.1. Postcomposing a nonharmonic biharmonic homothetic immersion defined in Theorem 4.5 to this map produces the required map. Finally, we can use biharmonic morphism define by the radial projection (see Proposition 3.2) to pull-back a nonharmonic biharmonic homothetic immersion defined in Theorem 4.5 to have a nonharmonic biharmonic map $\varphi : \mathbb{R}^4 \setminus \{0\} \to S^n$ for any $n \geq 3$. □

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